UDC 539.4

On Estimating a Constant Stress Life Test Model Using Time-Censored Data from the Linear Failure Rate Distribution

Ali A. Ismail[a,1] and M. M. Al-Harbi[b]

[a] Cairo University, Faculty of Economics & Political Science, Department of Statistics, Giza, Egypt

[b] Qassim University, College of Science and Arts, Department of Mathematics, Al-Rass, Saudi Arabia

[1] ali.ismail@cu.edu.eg (Ali A. Ismail)

УДК 539.4

Оценка модели долговечности при постоянном напряжении для цензурированных по времени данных с линейным распределением скорости разрушения

Али А. Исмаил[а], М. М. Аль-Харби[б]

[а] Каирский университет, Гиза, Египет

[б] Университет Кассима, Аль-Расс, Саудовская Аравия

Предложена модель частично ускоренных испытаний на циклическую долговечность с постоянным напряжением с использованием данных, подвергнутых цензурированию типа I с линейным распределением скорости разрушения. Получены точечные и интервальные оценки максимального правдоподобия параметров распределения и коэффициента ускорения. С целью повышения точности прогнозов расчет среднеквадратических погрешностей осуществлялся для образцов разных размеров. Для иллюстрации работоспособности модели выполнено моделирование тестовых задач с использованием метода Монте-Карло.

Ключевые слова: постоянное напряжение, линейное распределение скорости разрушения, максимальная вероятность, среднеквадратичная погрешность, цензурирование I типа.

Introduction. Today, with the high reliability of the materials or products, the life test under normal (use) condition will take a very long time and devour a large amount of money. In order to shorten the testing period, all or some of test units may be subjected to more severe conditions than use ones. Such a testing can be represented by fully accelerated life testing (FALT) or partially accelerated life testing (PALT). In FALT, the components/specimens are run under accelerated conditions but PALT combines both ordinary and accelerated life test. The major assumption in FALT is that the mathematical model relating the lifetime of units and stress is known or can be assumed. In some cases the life-stress relationship are not known and cannot even be assumed so that the data obtained from ALT can't be extrapolated to use condition. In such cases, PALT can be considered a good alternative technique to be used instead of FALT.

The main purpose of the FALT and PALT is to observe more failure data in a possible shortest time to draw statistical inference under use condition. FALT and PALT can be applied in various ways of stress, commonly used methods are step-stress and constant-stress. Under step-stress PALT, a test unit is first run at use condition and, if it doesn't fail then, it is run at accelerated condition until failure occurs or the observation is censored. But the constant-stress PALT run each unit either at use condition or accelerated condition only. Accelerated test stress can involve higher than usual temperature, voltage, pressure, load, humidity, etc., or some combination of them.

For an overview of constant-stress PALT (CSPALT) studies, one can refer, for example, to Ismail [1–3] and Ismail and Al Tamimi [4]. For readers interested in step-stress PALT, one can refer, for example, to Ismail [5] and Ismail and Al-Habardi [6, 7].

In this paper we seek to make estimation of parameters of the CSPALT model under time-censored data from the linear failure rate (LFR) distribution using the maximum likelihood (ML) method.

The constant-stress testing has several advantages: (i) it is easier to maintain a constant-stress level in most tests; (ii) accelerated test models for constant-stress are better developed for some materials and products; (iii) data analysis for reliability estimation is well developed. For more details, see [8].

This paper can be organized as follows. Section 1 introduces the LFR distribution as a failure time model and discusses the used test method. Section 2 presents the maximum likelihood estimates (MLEs) of the model parameters. Section 3 considers the confidence interval estimation of the model parameters. Section 4 presents simulation studies to illustrate the theoretical results.

1. **The Model and Test Method**.

1.1. ***The Linear Failure Rate Distribution***. A potential review of the LFR distribution as a lifetime-model in reliability and survival applications has been demonstrated well by Broadbent [9], Carbone et. al. [10], and Kodlin [11].

The probability density function of the LFR distribution for an item tested at use condition is given by

$$f_T(t) = (\alpha + \theta t)\exp\left[-\left(\alpha t + \frac{\theta t^2}{2}\right)\right], \qquad t > 0,\ \alpha > 0,\ \theta > 0,$$

where α is the shape parameter and θ is the scale parameter.

The corresponding reliability function is defined as

$$R(t) = \exp\left[-\left(\alpha t + \frac{\theta t^2}{2}\right)\right], \qquad \alpha > 0,\ \theta > 0.$$

Then, the hazard function is given by

$$h(t) = (\alpha + \theta t),$$

which is a linearly increasing function of time.

For an item tested at accelerated condition the probability density function of the LFR distribution can be given by

$$f_X(x) = (\alpha\beta + \theta\beta^2 x)\exp\left\{-\left(\alpha\beta x + \frac{\theta\beta^2 x^2}{2}\right)\right\}, \qquad x > 0,\ \alpha,\ \theta > 0,\ \beta > 1,$$

where the lifetime X of a test item run at accelerated condition is defined as $X = \beta^{-1}T$. Then the lifetime T of a test item run at use condition is obtained by $T = \beta X$, where $\beta > 1$ is the acceleration factor which is defined as

$$\beta = \frac{mean\ lifetime\ at\ use\ condition}{mean\ lifetime\ at\ accelerated\ condition}.$$

1.2. ***Constant-Stress PALT***. The test procedure can be described as follows:

(1) The total sample size of test items (n) is divided into two parts based on a pre-specified proportion π. The first part includes $n\pi$ items selected randomly to run under accelerated conditions, while the remaining items are allocated to run under use conditions.

(2) Each test item is run until the censoring time is reached or the item fails and the test condition is not changed. That is, the stress is still either design (use) stress or accelerated stress until the test is terminated.

It is assumed that:

(1) The lifetimes T_i, $i = 1, \ldots, n(1-\pi)$ of items allocated to run under use conditions, are independent and identically distributed (i.i.d.) random variables (r.v.'s).

(2) The lifetimes X_j, $j = 1, \ldots, n\pi$ of items allocated to run under accelerated conditions, are independent and identically distributed (i.i.d.) random variables (r.v.'s).

2. **MLEs of the Parameters**. In this section, the MLEs of the CSPALT model parameters using type-I censored data from the LFR distribution are obtained.

Now, let us define the indicator functions:

$$\delta_{u_i} = \begin{Bmatrix} 1, & t_i \le \eta, \quad i = 1, 2, \ldots, n(1-\pi) \\ 0, & o.w \end{Bmatrix}$$

and

$$\delta_{a_j} = \begin{Bmatrix} 1, & x_j \le \eta, \quad j = 1, 2, \ldots, n\pi \\ 0, & o.w \end{Bmatrix}.$$

Then the total likelihood function for (t_i, δ_{u_i}) and (x_j, δ_{a_j}) under CSPALT is given by

$$L(\underline{t}, \underline{x} | \alpha, \theta, \beta) = \prod_{i=1}^{n(1-\pi)} L_{u_i}(t_i, \delta_{u_i} | \alpha, \theta) \prod_{j=1}^{n\pi} L_{a_j}(x_j, \delta_{a_j} | \alpha, \theta, \beta).$$

That is,

$$L(\underline{t}, \underline{x} | \alpha,\theta,\beta) = \prod_{i=1}^{n(1-\pi)} [f_T(t_i)]^{\delta_{u_i}} \times [\bar{F}_T(\eta)]^{\bar{\delta}_{u_i}} \times \prod_{j=1}^{n\pi} [f_X(x_j)]^{\delta_{a_j}} \times [\bar{F}_X(\eta)]^{\bar{\delta}_{a_j}} =$$

$$= \prod_{i=1}^{n(1-\pi)} \left[(\alpha + \theta t_i) \exp\left\{ -\left(\alpha t_i + \frac{\theta t_i^2}{2} \right) \right\} \right]^{\delta_{u_i}} \times \left[\exp\left\{ -\left(\alpha\eta + \frac{\theta\eta^2}{2} \right) \right\} \right]^{\bar{\delta}_{u_i}} \times$$

$$\times \prod_{j=1}^{n(\pi)} \left[(\alpha\beta + \theta\beta^2 x_j) \exp\left\{ -\left(\alpha\beta x_j + \frac{\theta\beta^2 x_j^2}{2} \right) \right\} \right]^{\delta_{a_j}} \times \left[\exp\left\{ -\left(\alpha\beta\eta + \frac{\theta\beta^2\eta^2}{2} \right) \right\} \right]^{\bar{\delta}_{a_j}},$$

where $\bar{\delta}_{u_i} = 1 - \delta_{u_i}$ and $\bar{\delta}_{a_j} = 1 - \delta_{a_j}$.

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. So, the natural logarithm of the total likelihood function can be written as

$$\ln L = \sum_{i=1}^{n(1-\pi)} \delta_{u_i} \left[\ln(\alpha + \theta t_i) - \left(\alpha t_i + \frac{\theta t_i^2}{2} \right) + \left(\alpha\eta + \frac{\theta\eta^2}{2} \right) \right] +$$

$$+\sum_{j=1}^{n\pi}\delta_{a_j}\left[\ln(\alpha\beta+\theta\beta^2x_j)-\left(\alpha\beta x_j+\frac{\theta\beta^2x_j^2}{2}\right)+\left(\alpha\beta\eta+\frac{\theta\beta^2\eta^2}{2}\right)\right]-$$

$$-n(1-\pi)\left(\alpha\eta+\frac{\theta\eta^2}{2}\right)-n\pi\left(\alpha\beta\eta+\frac{\theta\beta^2\eta^2}{2}\right).$$

The first partial derivatives of the natural logarithm of the total likelihood function with respect to α, θ, and β are given by

$$\frac{\partial \ln L}{\partial\alpha}=\sum_{i=1}^{n(1-\pi)}\delta_{u_i}\left[\frac{1}{\alpha+\theta t_i}-t_i\right]+\sum_{j=1}^{n\pi}\delta_{a_j}\left[\frac{\beta}{\alpha\beta+\theta\beta^2x_j}-\beta x_j\right]+$$

$$+\eta(n_u-n(1-\pi))+\beta\eta(n_a-n\pi),$$

$$\frac{\partial \ln L}{\partial\theta}=\sum_{i=1}^{n(1-\pi)}\delta_{u_i}\left[\frac{t_i}{\alpha+\theta t_i}-\frac{t_i^2}{2}\right]+\sum_{j=1}^{n\pi}\delta_{a_j}\left[\frac{\beta^2x_j}{\alpha\beta+\theta\beta^2x_j}-\frac{\beta^2x_j^2}{2}\right]+$$

$$+\frac{\eta^2}{2}(n_u-n(1-\pi))+\frac{\beta^2\eta^2}{2}(n_a-n\pi),$$

$$\frac{\partial \ln L}{\partial\beta}=\sum_{j=1}^{n\pi}\delta_{a_j}\left[\frac{\alpha+2\theta\beta x_j}{\alpha\beta+\theta\beta^2x_j}-\alpha x_j-\theta\beta x_j^2\right]+(\alpha\eta+\theta\beta\eta^2)(n_a-n\pi).$$

where

$$n_u=\sum_{i=1}^{n(1-\pi)}\delta_{u_i},\qquad n_a=\sum_{j=1}^{n\pi}\delta_{a_j}.$$

The MLEs of α, θ, and β can be obtained by solving the system of the following equations:

$$\frac{\partial \ln L}{\partial\alpha}=0,\qquad \frac{\partial \ln L}{\partial\theta}=0,\qquad \frac{\partial \ln L}{\partial\beta}=0.$$

A closed form solution for these equations is hard to obtain. So, an iterative technique such as Newton–Raphson should be used to solve these equations, numerically.

Concerning the asymptotic variance-covariance matrix of the MLEs of α, θ, and β, it can be obtained by inverting the Fisher information matrix F. The asymptotic matrix F is composed of the negative second partial derivatives of the natural logarithm of the likelihood function evaluated at the MLEs. The second-order partial derivatives can be obtained as follows:

$$\frac{\partial^2 \ln L}{\partial\alpha^2}=-\sum_{i=1}^{n(1-\pi)}\delta_{u_i}\left[\frac{1}{(\alpha+\theta t_i)^2}\right]-\beta^2\sum_{j=1}^{n\pi}\delta_{a_j}\left[\frac{1}{(\alpha\beta+\theta\beta^2x_j)^2}\right],$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\theta}=\frac{\partial^2 \ln L}{\partial\theta\partial\alpha}=-\sum_{i=1}^{n(1-\pi)}\delta_{u_i}\left[\frac{t_i}{(\alpha+\theta t_i)^2}\right]-\beta^3\sum_{j=1}^{n\pi}\delta_{a_j}\left[\frac{x_j}{(\alpha\beta+\theta\beta^2x_j)^2}\right],$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\beta} = \frac{\partial^2 \ln L}{\partial\beta\partial\alpha} = \sum_{j=1}^{n\pi} \delta_{a_j} \left[\frac{-(\theta\beta^2 x_j)}{(\alpha\beta + \theta\beta^2 x_j)^2} - x_j \right] + \eta(n_a - n\pi),$$

$$\frac{\partial^2 \ln L}{\partial\theta^2} = -\sum_{i=1}^{n(1-\pi)} \delta_{u_i} \left[\frac{t_i^2}{(\alpha + \theta t_i)^2} \right] - \beta^4 \sum_{j=1}^{n\pi} \delta_{a_j} \left[\frac{x_j^2}{(\alpha\beta + \theta\beta^2 x_j)^2} \right],$$

$$\frac{\partial^2 \ln L}{\partial\theta\partial\beta} = \frac{\partial^2 \ln L}{\partial\beta\partial\theta} = \sum_{j=1}^{n\pi} \delta_{a_j} \left[\frac{(\alpha\beta^2 x_j)}{(\alpha\beta + \theta\beta^2 x_j)^2} - \beta x_j^2 \right] + \beta\eta^2 (n_a - n\pi),$$

$$\frac{\partial^2 \ln L}{\partial\beta^2} = \sum_{j=1}^{n\pi} \delta_{a_j} \left[\frac{-(\alpha^2 + 2\alpha\theta\beta x_j + 2(\theta\beta x_j)^2)}{(\alpha\beta + \theta\beta^2 x_j)^2} - \theta x_j^2 \right] + \theta\eta^2 (n_a - n\pi).$$

Hence, the asymptotic matrix F can be expressed by

$$F = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial^2\alpha^2} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\theta} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 \ln L}{\partial\theta\partial\alpha} & -\frac{\partial^2 \ln L}{\partial^2\theta^2} & -\frac{\partial^2 \ln L}{\partial\theta\partial\beta} \\ -\frac{\partial^2 \ln L}{\partial\beta\partial\alpha} & -\frac{\partial^2 \ln L}{\partial\beta\partial\theta} & -\frac{\partial^2 \ln L}{\partial^2\beta^2} \end{bmatrix}.$$

3. **Confidence Intervals of the Parameters**. The most common method to set confidence limits for the parameters is to use the large-sample normal distribution of the MLEs. For a large sample size, the maximum likelihood estimators are consistent and asymptotically normally distributed. Therefore, the two-sided approximate $100(1-\gamma)\%$ confidence limits for the model parameters can be obtained by

$$\hat{\alpha} \pm Z_{\gamma/2}\sqrt{F_{11}^{-1}}, \qquad \hat{\theta} \pm Z_{\gamma/2}\sqrt{F_{22}^{-1}}, \qquad \hat{\beta} \pm Z_{\gamma/2}\sqrt{F_{33}^{-1}},$$

where $Z_{\gamma/2}$ is the upper $(\gamma/2)$ percentile of the standard normal distribution.

4. **Simulation Studies**. In this section, simulation studies are prepared to demonstrate the theoretical results obtained in this paper. Tables 1 and 2 summarizes the results of ML estimates of the model parameters, estimated variances of the MLEs, mean squared errors (MSEs) and the average confidence interval width (CIW) based on 20,000 replications. Results of simulations studies provide insight into sampling behavior of estimators. As shown from Figs. 1 and 2, and the numerical results presented in Tables 1 and 2, the ML estimates approximate the true values of the parameters as the sample size n increases. Also, as shown from numerical results, the asymptotic variances and MSEs of the estimators decrease as the sample size n increases. In addition, the 95% CIW for the parameters decreases as the sample size n increases. That is, we obtain good estimates.

Conclusions. This paper considers the constant-stress PALT with time-censored data from the LFR distribution. Based on 20,000 replications, average values of the MLEs of the distribution parameters and the acceleration factor with MSEs and 95% CIW are obtained using different sizes of samples generated from LFR distribution. The maximum likelihood approach was used to estimate the model parameters. The estimates were obtained

Table 1

The ML Estimates of Parameters (β, α, θ) Set at (2, 3, 5), Respectively, Using 20,000 Replications, Given $\pi = 0.60$ and $\eta = 50$ for Different Sized Samples

n	Parameter	Estimate	Variance	MSEs	CIW
25	β	2.7920810	0.2587020	0.8197127	1.9663550
	α	3.6778810	0.9266840	1.2742290	3.6557720
	θ	5.5636810	0.1591740	2.1855450	1.4905760
30	β	2.7145270	0.2114490	0.6562398	1.7826160
	α	3.6190270	0.7562130	1.0363300	3.3204100
	θ	5.5235280	0.1407660	1.7984200	1.4136570
40	β	2.6472720	0.1568410	0.5052911	1.5407690
	α	3.5848720	0.5581330	0.7536431	2.8788620
	θ	5.4934080	0.1118290	1.2117770	1.2762460
50	β	2.5269340	0.1198690	0.3581571	1.3481250
	α	3.4110240	0.4213210	0.5735810	2.4987910
	θ	5.3434240	0.0968010	1.1130290	1.1893310
75	β	2.3703680	0.0747850	0.2046902	1.065951
	α	3.2806720	0.2672380	0.4690595	1.989458
	θ	5.2102720	0.0684100	1.0308520	1.002973
100	β	2.3001520	0.0544650	0.1544027	0.9099452
	α	3.1776120	0.1923980	0.4192237	1.6865700
	θ	5.1032120	0.0532170	1.0129360	0.8851347

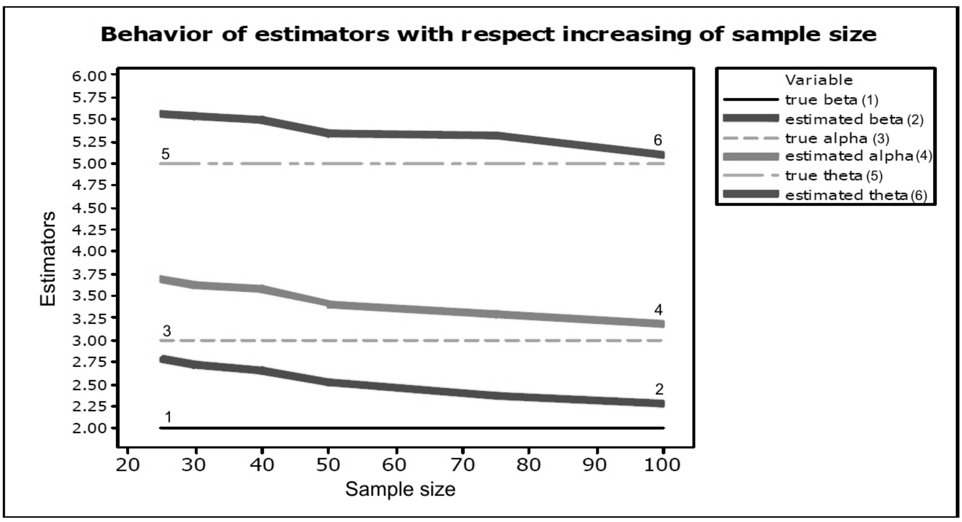

Fig. 1. Behavior of ML estimators when (β, α, θ) set at (2, 3, 5), respectively, according to the increase in the sample size.

numerically using the Newton–Raphson technique. The behavior of the ML estimates seems to be good as shown from the numerical results presented in both Tables 1 and 2. Also, both Figs. 1 and 2 show that the obtained estimates are consistent. The present study can be extended to provide the reliability analysis via the Bayesian approach.

T a b l e 2

The ML Estimates of Parameters (β, α, θ) Set at (1.5, 0.75, 0.6), Respectively, Using 20,000 Replications, Given $\pi = 0.60$ and $\eta = 50$ and for Different Sized Samples

n	Parameter	Estimate	Variance	MSEs	CIW
25	β	1.9782690	0.0688237	0.2497827	1.0070310
	α	0.9782695	0.1140310	0.0731479	1.3090040
	θ	0.9763770	0.0746381	0.1641288	1.0503820
30	β	1.8161060	0.0588099	0.1272452	0.93755
	α	0.9024745	0.0838522	0.05673902	1.11198
	θ	0.8832815	0.0667339	0.112314	0.9863347
40	β	1.701863	0.0463648	0.0740529	0.8366053
	α	0.8421055	0.0569019	0.0566578	0.9042282
	θ	0.7486105	0.0508923	0.0600419	0.8523467
50	β	1.6510100	0.0377266	0.0542071	0.7564447
	α	0.8182300	0.0432931	0.0545797	0.7862353
	θ	0.7160400	0.0416022	0.0515632	0.7691459
75	β	1.5745250	0.0252371	0.0312259	0.6201672
	α	0.8058870	0.0281109	0.0541501	0.6323550
	θ	0.6855785	0.0287435	0.0440897	0.6389501
100	β	1.5540690	0.0194176	0.0279133	0.5444356
	α	0.7846060	0.0200449	0.0518522	0.5330381
	θ	0.6462835	0.0216717	0.0363211	0.5542264

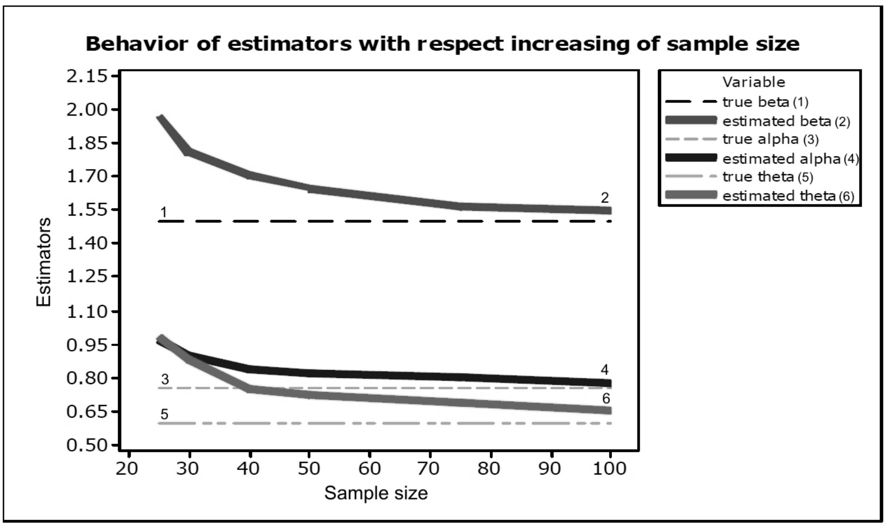

Fig. 2. Behavior of ML estimators when (β, α, θ) set at (1.5, 0.75, 0.6), respectively, according to the increase in the sample size.

Р е з ю м е

Запропоновано модель частково прискорених випробувань на циклічну довговічність з постійною напругою з використанням даних, які було піддано цензуруванню типу I із лінійним розподілом швидкості руйнування. Отримано точкові й інтервальні оцінки максимальної правдоподібності параметрів розподілу і коефіцієнта прискорення. Із метою підвищення точності прогнозів розрахунок середньоквадратичних похибок проводився для зразків різних розмірів. Для ілюстрації роботоздатності моделі виконано моделювання тестових задач із використанням методу Монте-Карло.

1. Ali A. Ismail, "Estimating the generalized exponential distribution parameters and the acceleration factor under constant-stress partially accelerated life testing with type-II censoring," *Strength Mater.*, **45**, No. 6, 693–702 (2013).
2. Ali A. Ismail, "On designing constant-stress partially accelerated life tests under time-censoring," *Strength Mater.*, **46**, No. 1, 132–139 (2014).
3. Ali A. Ismail, "Bayesian estimation under constant-stress partially accelerated life test for Pareto distribution with type-I censoring," *Strength Mater.*, **47**, No. 4, 633–641 (2015).
4. Ali A. Ismail and A. Al Tamimi, "Optimum constant-stress partially accelerated life test plans using type-I censored data from the inverse Weibull distribution," *Strength Mater.*, **49**, No. 6, 847–855 (2017).
5. Ali A. Ismail, "Planning step-stress life tests for the generalized Rayleigh distribution under progressive type-II censoring with binomial removals," *Strength Mater.*, **49**, No. 2, 292–306 (2017).
6. Ali A. Ismail and K. Al-Habardi, "On designing time-censored step-stress life test for the Burr type-XII distribution," *Strength Mater.*, **49**, No. 5, 699–709 (2017).
7. Ali A. Ismail and K. Al-Habardi, "Optimum plans of step-stress life tests using failure-censored data form Burr type-XII distribution," *Strength Mater.*, **50**, No. 4, 674–685 (2018).
8. W. Nelson, *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis*, John Wiley & Sons, New York (1990).
9. S. Broadbent, "Simple mortality rates," *J. R. Stat. Soc. C-Appl.*, **7**, No. 2, 86–95 (1958).
10. P. P. Carbone, L. E. Kellerhouse, and E. A. Gehan, "Plasmacytic myeloma: a study of the relationship of survival to various clinical manifestations and anomalous protein type in 112 patients," *Am. J. Med.*, **42**, No. 6, 937–948 (1967).
11. D. Kodlin, "A new response time distribution," *Biometrics*, **23**, No. 2, 227–239 (1967).

Received 05. 02. 2018