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Theoretical Construction of Initial and Subsequent Yield Surfaces for Isotropic Strain-Hardening Elastoplastic Materials of the Differential Type

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Теоретическое построение начальной и последующих поверхностей нагружения изотропных упрочняющихся упругопластических материалов дифференциального типа

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В рамках теории бесконечно малых деформаций предложен новый подход к теоретическому построению начальной и последующей поверхностей нагружения для некоторого класса простых по Ноллу начально твердых упрочняющихся упругопластических материалов дифференциального типа сложности 1. При этом поверхности нагружения строятся без принятия каких-либо дополнительных предположений, для всего рассмотренного класса материалов начальная поверхность имеет форму Губера–Мизеса, а последующие поверхности в общем случае подвергаются расширению, смещению и изменению формы. В частных случаях уравнение последующей поверхности нагружения совпадает с уравнениями известных теорий пластичности с изотропно-кинематическим и кинематическим упрочнением. Установлено, что для начальных поверхностей нагружения и регулярных точек последующих поверхностей нагружения выполняется ассоциированный закон течения. При принятых определениях активного нагружения, разгрузки и нейтрального нагружения для начальной и гладких выпуклых последующих поверхностей нагружения выполняется постулат Друккера. Обосновано проведение опытов, необходимых для конкретизации полученных зависимостей.

Ключевые слова: простой по Ноллу упрочняющийся упругопластический материал дифференциального типа сложности 1, бесконечно малые деформации, изотропия, поверхность нагружения.

In our earlier studies [1, 2], using rational continuum mechanics approaches, a mathematical theory was developed for a strict construction and specialization of constitutive equations for simple (in Noll's sense) isotropic strain-hardening elastoplastic materials of the differential type of complexity n as the most important representatives of the materials with the infinitesimal memory of the path shape (the path shape memory on an arbitrarily small interval of the "past"). The strains were assumed to be finite. The hierarchy of the constitutive relations was constructed according to the level of complexity of the material response to deformation.

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In papers [1, 3], the method for specialization of the previously constructed physical equations [1, 2] is developed within the theory of infinitesimal strains. With this method, a number of constitutive relations are derived. For $n = 1$, the conditions for the existence of a loading surface are established.

In [3], a number of assumptions are made for the specialization of constitutive relations.

Assumption 1. We assume that in the construction of the constitutive relations for simple, in Noll's sense, isotropic strain-hardening elastoplastic bodies of the differential type of a different complexity, a specific character of the tensor space structure related to the tensor product can be neglected.

Assumption 2. We assume that under the elastoplastic deformation, the stress deviator is independent of the first invariants and the mean stress is independent of the deviatoric components of the kinematic infinitesimal strain tensors.

Assumption 3. We assume that the prehistory of variation of elastic components of the strain tensor does not influence the stresses in the elastoplastic material.

By the prehistory of variation of elastic components of the strain tensor one should mean the whole history of variation of the elastic component of the strain tensor except for its value at the end of the deformation process.

Assumptions 1 and 2 impose constraints on the material properties and, as noted in [3], they are verified experimentally for a number of materials.

For strain-hardening elastoplastic materials of the differential type of complexity 1, which obey assumptions 1–3, a constitutive relation is constructed that is valid for the general case of active deformation and has the following form [3]:

$$s = \eta_2 e + \eta_3 \hat{e}_1^p, \tag{1}$$

where η_2 and η_3 depend on invariants

$$tr\,(e)^2, \qquad tr\,(e\hat{e}_1^p), \tag{2}$$

s, e, and $\hat{e}_1^p = de^p / d\xi^p$ are the stress, total strain, and plastic strain rate deviators, respectively, e^p is the plastic strain deviator, and ξ^p is the arc length of the path of the plastic strain deviator.

According to [3], equation (1) is a parametric representation of the loading surface equation. To analyze in greater detail the loading surface properties (such as closure, convexity, smoothness, the associated flow rule observance or violation, etc.), the specialization of functions η_2 and η_3 – to a greater or lesser extent – is required.

The purpose of this work is to develop an approach to the construction of the initial and subsequent loading surfaces based on Eq. (1) without specializing functions η_2 and η_3.

As shown by the analysis, it follows from the general properties of the initially solid strain-hardening elastoplastic material (hereinafter referred to as the strain-hardening elastoplastic material) [4] that the path (history of variation) of the plastic strain during active deformation fully determines the stresses at the end of the deformation process. Based on the aforementioned, a conclusion can be made that assumption 3 stems from the general properties of strain-hardening elastoplastic materials and can be reformulated as follows: in setting the history of variation of plastic strains, the prehistory of variation of elastic components of the strain tensor can be neglected in determining the stresses in the strain-hardening elastoplastic material. Moreover, in constructing constitutive relations for such materials we can admit that, in case of setting the history of variation of plastic strains, the history of variation of elastic components of the strain tensor can be neglected in determining the stresses in the strain-hardening elastoplastic material.

If the last corollary that proceeds from the general properties of the strain-hardening elastoplastic materials considered proves to be valid, Eq. (1) takes the form:

$$s = \hat{\eta}_2^p e^p + \hat{\eta}_3^p \hat{e}_1^p, \tag{3}$$

where coefficients $\hat{\eta}_2^p$ and $\hat{\eta}_3^p$ are certain functions of invariants

$$tr\,(e^p)^2, \qquad tr(e^p \hat{e}_1^p). \tag{4}$$

As follows from [3], Eq. (3) can be written in the vector space. In this case, the stress, plastic strain and plastic strain rate deviators can be substituted by relevant vectors.

Let us write the vector representation of Eq. (3) in the Il'yushin space [5]

$$\vec{\sigma} = \vec{\eta}_2^p \vec{E}^p + \vec{\eta}_3^p \vec{E}_1^p, \tag{5}$$

where $\vec{\sigma}$, $\vec{E}^p$, and $\vec{E}_1^p = d\vec{E}^p \big/ d\xi^p$ are the stress, plastic strain, and plastic strain rate vectors, respectively, $\vec{\eta}_2^p$ and $\vec{\eta}_3^p$, in view of (4), depend on the modulus of the plastic strain vector and the angle between the plastic strain vector and the plastic strain "rate" vector. At every point of the active deformation path, coefficients $\hat{\eta}_2^p$ and $\hat{\eta}_3^p$ correspond to coefficients $\vec{\eta}_2^p$ and $\vec{\eta}_3^p$, respectively.

At the beginning of the active deformation process, the plastic strain is equal to zero. In this case, Eq. (3) takes the form

$$s = \hat{\eta}_{30}^p \hat{e}_1^p, \tag{6}$$

where, in view of (4), $\hat{\eta}_{30}^p$ is the positive quantity that does not depend on the type of the stress (strain) state.

Taking into account the fact that the stress deviator in the active process of uniaxial tension at the beginning of plastic deformation equals to s_{T0}, where s_{T0} is the stress deviator at the yield point under uniaxial tension, equation (6) can be rewritten as

$$s = \hat{\eta}^{p}_{30} \hat{e}^{p}_{1} = s_{T0}. \tag{7}$$

Let us bring Eq. (7) to the invariant form. To do so, let us square it, take the trace of the obtained equation, multiply all its members by 3/2 for convenience, and write it in the coordinate form as follows:

$$\frac{3}{2} s_{ij} s_{ij} = \frac{3}{2} (\hat{\eta}^{p}_{30})^2 = \frac{3}{2} s_{ij_{T0}} s_{ij_{T0}} = \sigma^2_{T0}, \tag{8}$$

where σ_{T0} is the initial yield strength under uniaxial tension. In writing Eq. (8), we took into account the fact that $\hat{e}^{p}_{1}$ is a normalized deviator.

Relation (8) corresponds to the equation of the initial loading surface, which can be represented as

$$f_{in}(\sigma_{ij}) = \frac{3}{2} s_{ij} s_{ij} - \sigma^2_{T0} = \sigma^2_i - \sigma^2_{T0} = 0, \tag{9}$$

where σ_{ij} and $\sigma_i = \sqrt{(3/2) s_{ij} s_{ij}}$ are the Cauchy stress tensor and the stress intensity, respectively.

Under assumptions adopted in the construction of Eq. (9), the initial loading surface in the three-dimensional space of principal stresses represents a circular cylinder (the Huber–Mises cylinder) – a regular (smooth), continuous, convex surface with the radius equal to $\sqrt{2/3}\, \sigma_{T0}$.

Given the validity of Eq. (9), we can show [6] that

$$\frac{\partial f_{in}}{\partial \sigma_{ij}} = \frac{\partial f_{in}}{\partial s_{ij}} = 3 s_{ij}, \qquad \text{and} \qquad \frac{\partial f_{in}}{\partial \sigma_{ij}} d\sigma_{ij} = \frac{\partial f_{in}}{\partial s_{ij}} ds_{ij}. \tag{10}$$

Based on Eqs. (6), (8), and (9), we write in the coordinate form

$$de^{p}_{ij} = \sqrt{\frac{3}{2}} \frac{d\xi^{p}}{\sigma_i} s_{ij}. \tag{11}$$

In view of (10), Eq. (11) expresses the flow rule associated to (9)

$$de^{p}_{ij} = d\lambda \frac{\partial f}{\partial \sigma_{ij}} = d\lambda \frac{\partial f}{\partial s_{ij}} = \sqrt{\frac{3}{2}} \frac{d\xi^{p}}{\sigma_i} s_{ij}, \tag{12}$$

where, as follows from (12), $d\lambda = \frac{1}{\sqrt{6}} \frac{d\xi^{p}}{\sigma_i} > 0.$

Using the concepts of the active loading, unloading and neutral loading [7] and taking into account (10), we obtain for the initial loading surface (9) under active loading, unloading and neutral loading, respectively [6]

$$\frac{\partial f_{in}}{\partial s_{ij}} ds_{ij} > 0, \quad f_{in} = 0, \quad d\lambda > 0, \tag{13}$$

$$\frac{\partial f_{in}}{\partial s_{ij}} ds_{ij} < 0, \quad f_{in} = 0, \quad d\lambda = 0, \tag{14}$$

$$\frac{\partial f_{in}}{\partial s_{ij}} ds_{ij} = 0, \quad f_{in} = 0, \quad d\lambda = 0. \tag{15}$$

Given the validity of Eqs. (12)–(15), the Drucker's postulate is fulfilled for the initial Huber–Mises loading surface.

In constructing subsequent loading surfaces when plastic strains are not equal to zero, we go back to Eq. (3), whose coefficients depend on invariants (4), and which has form (5) in the Il'yushin vector space.

Let us introduce the concept of the active stress deviator [8, 9], by which one should mean the component of the stress deviator decomposition in the direction of the line tangent to the plastic strain deviator path. Then we can write from Eq. (3)

$$s^a = s - \hat{\eta}_2^p e^p = \hat{\eta}_3^p \hat{e}_1^p , \tag{16}$$

where s^a is the active stress deviator.

It follows from Eq. (16) that the direction tensors, strain mode angles and Lode–Nadai parameters of the active stress deviator and $\hat{e}_1^p$ in the arbitrary active-deformation process, within the limits of validity of relation (3), coincide. Since the $\hat{e}_1^p$ deviator is normalized, then $\left| s^a \right| = \left| \hat{\eta}_3^p \right|$.

By squaring (16), taking the trace of th obtained equation, multiplying its right- and left-hand sides by 3/2 for convenience, we can write it in the coordinate form

$$\frac{3}{2} s_{ij}^a s_{ij}^a = \frac{3}{2}(s_{ij} - \hat{\eta}_2^p e_{ij}^p)(s_{ij} - \hat{\eta}_2^p e_{ij}^p) = \frac{3}{2}(\hat{\eta}_3^p)^2 . \tag{17}$$

Equation (17) represents the equation of the subsequent loading surface, which can be rewritten as follows:

$$f(\sigma_{ij}) = (\sigma_i^a)^2 - \frac{3}{2}(\hat{\eta}_3^p)^2 = \frac{3}{2} s_{ij}^a s_{ij}^a - \frac{3}{2}(\hat{\eta}_3^p)^2 = 0, \tag{18}$$

where $\sigma_i^a = \sqrt{(3/2) s_{ij}^a s_{ij}^a}$ is the active stress intensity.

In the case of validity of Eq. (18), using the approach described in [6], we can show that

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial s_{ij}} = 3(s_{ij} - \eta_2^p e_{ij}^p) = 3 s_{ij}^a, \qquad \text{and} \qquad \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = \frac{\partial f}{\partial s_{ij}} ds_{ij} . \tag{19}$$

As follows from Eqs. (16)–(18) in the coordinate form,

$$de_{ij}^p = \sqrt{\frac{3}{2}} \frac{d\xi^p}{\sigma_i^a} s_{ij}^a. \tag{20}$$

In view of (19), Eq. (20) expresses the flow rule associated to (18)

$$de_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}} = d\lambda \frac{\partial f}{\partial s_{ij}} = \sqrt{\frac{3}{2}} \frac{d\xi^p}{\sigma_i^a} s_{ij}^a, \tag{21}$$

where, as follows from (21), $d\lambda = \frac{1}{\sqrt{6}} \frac{d\xi^p}{\sigma_i^a} > 0.$

Using the concepts of the active loading, unloading and neutral loading for regular points of the loading surface, we derive that for the active loading, unloading and neutral loading the following equations are true, respectively

$$\frac{\partial f}{\partial s_{ij}} ds_{ij} > 0, \quad f = 0, \quad d\lambda > 0, \tag{22}$$

$$\frac{\partial f}{\partial s_{ij}} ds_{ij} < 0, \quad f = 0, \quad d\lambda = 0, \tag{23}$$

$$\frac{\partial f}{\partial s_{ij}} ds_{ij} = 0, \quad f = 0, \quad d\lambda = 0. \tag{24}$$

Given the validity of Eqs. (21)–(24), the Drucker's postulate is fulfilled for regular points of the subsequent loading surface of convex shape.

Following the procedure described in [6], we can show that, for the initial and subsequent loading surfaces represented by (9) and (21), the plastic change of volume equals to zero, i.e., the material is plastically incompressible.

Let us point out that no other assumption was taken into account in the construction of Eqs. (9) and (18) based on Eq. (1).

We will call $\hat{\eta}_2^p e_{ij}^p$ and $\hat{\eta}_3^p$ in (17) the evolution parameters of the subsequent loading surface. In view of (4) and the vector representation of the deviatoric space, $\hat{\eta}_2^p$ and $\hat{\eta}_3^p$ depend on the plastic strain deviator (vector) modulus, angle between the plastic strain vector and the plastic strain "rate" vector. Equation (17) allows modeling the translation, expansion and distortion of the subsequent loading surface in the active deformation process. Here, the distortion of the subsequent loading surface is described by the angle between the plastic strain vector and the plastic strain rate vector. As we know from [6, 10], this kind of evolution of the subsequent loading surface and the initial surface of the Huber–Mises type are characteristic of a number of elastoplastic materials.

Let us note that the plastic strain deviator modulus does not describe the effect caused by the path shape of the plastic strain deviator on $\hat{\eta}_2^p$ and $\hat{\eta}_3^p$ in Eq. (17). Therefore, these coefficients can be specified based on uniaxial tests with the additional construction of subsequent loading surfaces for every plastic strain deviator modulus within the studied plastic strain range in accordance with the technique presented in [10], for instance.

Let us consider some special cases of Eq. (17).

1) We assume that for a certain class of materials considered here coefficient $\hat{\eta}_2^p$ does not depend on the angle between the plastic strain vector and the plastic strain "rate" vector. At the same time, a distortion of the subsequent loading surface can be described by the evolution parameter $\hat{\eta}_3^p$ that depends on the above angle.

2) We assume that $\hat{\eta}_2^p$ and $\hat{\eta}_3^p$ do not depend on the angle between the plastic strain vector and the plastic strain "rate" vector. Then the equation of the subsequent loading surface takes the Huber–Mises form, $\hat{\eta}_2^p$ and $\hat{\eta}_3^p$ are determined by the plastic strain deviator modulus. In this case, the equation of the subsequent loading surface can be specified based on uniaxial tests performed in accordance with the technique presented in [6], which was offered for the Kadashevich–Novozhilov theory of plasticity with isotropic-kinematic hardening [8]. In [11], the translation of the subsequent loading surface ($\hat{\eta}_2^p e_{ij}^p$) is modeled by the dependence of the plastic strain deviator coefficient on the plastic strain deviator modulus.

3) In special cases such as $\sqrt{3/2}\,\hat{\eta}_3^p = \sigma_{T0} = \text{const}$ and $\hat{\eta}_2^p = c = \text{const}$, Eq. (17) corresponds to the equation of the subsequent loading surface from the Ishlinsky–Prager theory [12, 13].

The equation of the initial loading surface for special classes of materials considered here will not change.

The main distinction of the proposed approach from the well-known approaches [6–9, 11–14, and others] is that the loading surface type is not postulated, as well as the constitutive equation based on which it is constructed. The initial and subsequent loading surfaces, as well as the expression for the plastic strain increment deviator, are constructed strictly on the basis of the earlier obtained constitutive relation that is valid for the active loading processes.

Within the theory of infinitesimal strains, a new approach is proposed for the theoretical construction of the initial and subsequent loading surfaces for simple in Noll's sense initially solid strain-hardening materials of the differential type of complexity 1. The loading surfaces are constructed without making additional assumptions. For the whole class of materials considered, the initial surface has the Huber–Mises form, the subsequent surface undergoes expansion, translation and distortion in the general case. In special cases, the equation of the subsequent loading surface corresponds to the equations of the well-known theories of plasticity with the isotropic-kinematic and kinematic hardening. It is established that the associated flow rule is fulfilled for the initial loading surfaces and regular points of the subsequent loading surfaces. With the accepted definitions of the

active loading, unloading and neutral loading, the Drucker postulate is fulfilled for the initial surface and subsequent surfaces of smooth convex shape. The experiments necessary for the concrete definition of the derived dependences are verified.

Резюме

У рамках теорії малих деформацій запропоновано новий підхід до теоретичної побудови початкової і наступної поверхонь навантаження для деякого класу простих за Ноллом початково твердих зміцнюваних пружно-пластичних матеріалів диференціального типу складності 1. При цьому поверхні навантаження будуються без прийняття будь-яких додаткових припущень, для всього класу матеріалів, що розглядаються, початкова поверхня має форму Губера–Мізеса, а наступні поверхні в загальному випадку зазнають розширення, зміщення і зміни форми. В окремих випадках рівняння наступної поверхні навантаження збігається з рівнянням відомих теорій пластичності з ізотропно-кінематичним і кінематичним зміцненням. Установлено, що для початкових поверхонь навантаження і регулярних точок наступних поверхонь навантаження виконується асоційований закон текучості. При прийнятих визначеннях активного навантаження, розвантаження і нейтрального навантаження для початкової і гладких опуклих наступних поверхонь навантаження виконується постулат Друккера. Обґрунтовано проведення випробувань, що необхідні для конкретизації отриманих залежностей.

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